The Financial Crisis: Our Share of the Blame

Don McLeish
University of Waterloo

Abstract. The credit crisis has been variously blamed on compensation practices and greed, "quants", mortgage brokers, mathematical models used to price credit derivatives, mathematics, statistics, physicists, the rating agencies, defects in risk measurement models and leverage and lax capital requirements, to mention only a fraction. What is the statistican's share of the blame? To what extent are the statistical models used in risk measurement and credit risk and derivative price modeling culpable? In this talk I will give a short review of risk measurement and models for credit default risk. Potential causes of instability in the models and possible remedies are discussed.

Key Words: financial networks, credit crisis, risk measurement

1. Introduction

I begin with a quote[1] from Warren Buffet, C.E.O. of Berkshire Hathaway, written while the echoes of the bursting dot-com bubble were still ringing in the markets in 2001: “You only find out who is swimming naked when the tide goes out”. These were heady days when new invisible clothing was worn by emperor geeks in the dot-com businesses and their overheated investors. The financial community seemed by contrast neo-Victorian prudes. Unfortunately, in the past three years, the financial tide went out a lot further than most of our nightmares envisioned revealing that the best dressed of the ill-conditioned bankers (some Canadian) were in thongs. The rest of the skinny dippers had parked their flimsy risk-management tools on shore and declared the tide a rare event of little consequence, just before being swept out to sea.

Leaving this ill-considered nude-banker image aside, there followed two years of discussion about who was to blame, for the tide, for the nudity, and whose responsibility it is to ensure that bankers remain conservative in risk profile and dress. The list of scapegoats continues: from everpresent over-optimism and greed, reluctant government regulators, unscrupulous fraudsters, naive and poorly educated investors, through Milton Friedman and the Efficient Market Hypothesis[2]. As the science devoted to the study and quantification of uncertainties, where does our discipline stand in this role call of the negligent and culpable?

[1] http://www.berkshirehathaway.com/2001ar/2001letter.html

[2] The grand illusion: How efficient-market theory has been proved both wrong and right. *The Economist*, March 5, 2009. http://www.economist.com/finance/PrinterFriendly.cfm?story_id=13240822

Our models and methods and their practitioners have figured prominently both in the bubble and the ensuing bust, and this has not escaped notice of the press. Quants[3] have been variously vilified for their "complex mathematical models" many of which are straightforward by-products of normal models, with David Li, a University of Waterloo graduate, and the copula model he proposed for pricing collateralized debt obligations near the top of this list of quant reprobates.

My primordial reaction to the question "whose fault is it anyway" is much like yours, I suspect. It is the fault of (executive) compensation schemes, poorly educated consumers of financial products, self-interested rating agencies, mortgage brokers and investment banks, the Bernie Madoffs of the world and those unwilling or unable to unmask them, basically everyone but us. My purpose here will be to give this question a second look, specifically with "us", by which I mean statistical models, in mind.

Why is it natural that we should have a stake in this issue beyond our failing retirement savings plans and swollen unemployment statistics? Statisticians offer tools and experience, with random variation, model and parameter testing, robustness, dependence and multivariate modelling, as well as a well-grounded understanding that a distribution is more complex than one or two constants, quantiles or moments can possibly do justice to. No simple constraint on a scalar measure of a risk distribution, whether quantile or conditional moment can possibly militate against gaming, unless preceded by a universal decree that *all distributions are created equal* and moreover all compelled to be (multivariate) normal.

The total GDP of the U.S. is around 14 trillion dollars and each of the last two quarters it fell at around a 6% annual rate, or about a trillion dollars erased from GDP in a year, not to mention the trillions erased globally. When this mess began in 2006, there was a total notional of around 560 billion in CDOs outstanding, about twice what it had been four years earlier. If one third of these experienced default, that erases a mere 200 billion from GDP. How could such a small tail wag such a monstrous dog?

I operate under the dual constraints of limitations of time and ever-active active constraint of lack of expertise, so I will concentrate on features of financial and risk measurement models which I think have contributed to the problem more than to the solution. Billy Wilder said[4], "*Hindsight is always 20-20*". Closer to home, Charles Morris writes[5] *"intellectuals are reliable lagging indicators, near-infallible guides to what use to be true."*. I do not work out of the Vatican and make no claim to infallibility, even in retrospect, but nevertheless let me begin to identify aspects of the statistical models used in finance which have contributed to the crisis we now enjoy.

First, some basic definitions relevant to the discussion. *Credit risk* is the distribution of loss due to failure of a financial agreement. A *Credit derivative* is a security which allows the transfer of credit risk from one party to another. The two most common examples of credit derivatives are the CDS and the CDO. A *Credit Default Swap* (CDS) is an agreement whereby, in the event of default, the protection seller compensates the protection buyer for loss. In return, the buyer makes (quarterly) payments of the swap spread. According to the BIS quarterly review (June 2009), there was globally a notional of about $42 trillion in

[3]" I think it is pure scapegoating to blame the models and mathematicians. It's like blaming World War II on the German language" Marco Avellaneda in "Feeling financial pain? First, blame all the Scientists" *USA Today*, Oct 10, 2008.

[4]Billy Wilder (b. 1906), U.S. film director. Quoted *Wit and Wisdom of the Moviemakers*, ch. 7, ed. John Robert Columbo (1979)

[5]Charles Morris in *The Trillion Dollar Meltdown,*

CDS contracts in December 2008, down about 25% from its high in Dec. 2007, but to put this in perspective, still three times the annual G.D.P. of the U.S.A.

Let me explain a CDS in more graphic terms. Let us suppose that I am able to purchase 1 million dollars in fire insurance, not only on my own house, but on that of my neighbor. I have no ownership stake in my neighbor's house, indeed I do not like my neighbor very much. Moreover it is perfectly legal to light small fires in the basement or yard of my neighbor or encourage others to do so, and hope that the fire spreads. This is, in effect, a CDS. Of course fire insurance companies have the luxury of diversification...arsonists rarely carry their rampages across the country or the globe. Not so with a CDS, since the near independence of defaults, assumed under "normal" conditions, promptly evaporates in crisis, or in the very circumstance that the CDS will be needed.

AIG, the largest insurer in the world was a huge writer of credit default swaps with notional exposure at the end of September 2008 of $372.3bn[6]. The required collateral increased not only with the probability of default increased but as AIG's credit rating fell from a relatively good rating (AA-). In September 2008, the rating agencies cut the credit rating, AIG's counterparties demanded more collateral, and bankruptcy was averted only by government bailout.

Collateralized debt obligations (CDOs) are portfolios of fixed-income assets or CDSs (synthetic CDOs) typically divided into different tranches: senior tranches (AAA), mezzanine tranches (AA to BB), and equity tranches (unrated). The holders of the more senior tranches have first claim to any income from the portfolio (but at a lower return) than those holding the equity and mezzanine tranches. This highly profitable process of assembling a portfolio of assets like bonds or mortgages, slicing and dicing it into tranches and then selling these tranches is called securitization. It was carried out with collaboration from the rating agencies who, with the benefit of statistical models, then assigned these investment rating categories which systematically underestimated their inherent risk. The London-based AIG Financial Products sold $71.6bn[6] in collateralized debt obligations at the end of September 2008.

2. The Measurement of Risk and Reward

The corporate world regularly rewards executives and traders for "performance" with bonuses and executive stock options, products with highly asymmetric payoffs. AIG, which has put over 100 billion dollars in federal bailout money at risk[7], awarded in excess of $1 billion in retention payments and performance bonuses to employees across the organization and $165 million to employees of its financial products unit (responsible for most of its losses).[8] A major player in the subprime mortgage market, Countrywide financial's CEO Angelo Mozilio made over 140 million dollars in the sale of stock largely obtained through executive options[9]. Individual traders like Brian Hunter at Amaranth, Meriwether at Long Term Capital, Barings Bank's Nick Leeson, Jérôme Kerviel at Société Générale as well as corporations like Enron, Lehman Bros., and AIG learned that participating in high-risk ventures was richly rewarded, and that the risk profile could be manipulated so any constraints on risk could either be ignored or gamed away.

[6]Cris Sholto Heaton in *AIG's $170bn bail-out*, MoneyWeek, Dec 19, 2008

[7]"AIG Breakup is Fee Bonanza" *The Wall Street Journal*, Aug. 6, 2009

[8]Reuters: Thu Jul 9, 2009 http://www.reuters.com/article/businessNews/idUSTRE56906T20090710

[9]New York Times, Thursday, July 16, 2009

The most firmly established measures of risk in practice are variance, semi-variance (which uses the departures on one side only) and Value at Risk, or VaR, which is simply a quantile of the loss distribution over a specific time frame. VaR is given special prominence in the Basel II accord [10]. The pitfalls associated with summarizing a distribution using a single moment such as variance is well-understood by statisticians. The defects in using VaR as the primary risk measurement tool, though public knowledge for at least the past 10 years[11], are less-well understood. Academic articles about the inconsistencies associated with risk measures like VaR go back at least 11 years. Nassim Nicholas Taleb, the best-selling author of "The Black Swan", describes VaR as an "air bag that works all the time, except when you have a car accident", and has crusaded against VaR for more than a decade[12]. Joe Nocera, a *New York Times* columnist, has written several articles pointing out the shortcomings of VaR and its role in the current risk assessment crisis. VaR specifies a quantile but neither specifies nor controls what happens to the losses when that quantile is exceeded. "The fact that you are not likely to lose more than a certain amount 99 percent of the time tells you absolutely nothing about what could happen the other 1 percent of the time. You could lose \$51 million instead of \$50 million — no big deal. That happens two or three times a year, and no one blinks an eye. You could also lose billions and go out of business. VaR has no way of measuring which it will be."[11] Even the largest pension fund in Quebec, the Caisse de Dépôt et Placement du Québec, was misled to the tune of 40 billion or 25% of its value relying on risk measures like VaR[13]. It is well-understood that the incentive system has encouraged risk-taking[14], and the following example is simply another indication that with conventional risk-measures, this remains the case.

Example: We provide a simple example to show the ease with which a distribution can be modified to exploit the deficiencies in VaR. Suppose that the one-period loss L has the following probability density function

$$f(z) = \begin{cases} \frac{1}{\sigma}\varphi(\frac{z-\mu}{\sigma}) & \text{if } z \le c = \mu + \sigma z_p \\ \frac{p}{\beta}e^{-\frac{z-c}{\beta}} & \text{if } z > c \end{cases} .$$

where φ and Φ are the standard normal pdf and cdf respectively and $z_p = \Phi^{-1}(1-p)$. Clearly c has been chosen so that whatever the values of the parameters, $VaR_{1-p} = c$, which we hold constant. There is nothing magic about either the normal center of the distribution or the exponential tail. Any pdf generated as a spline would result in a similar conclusion. Suppose the parameters μ and p are fixed, with p very small. By various investments, we assume that it is possible to modify the parameter β in the tail of the

[10] Basel Committee on Banking Supervision International Convergence of Capital Measurement and Capital Standards. A Revised Framework, November 2005: "No particular type of VaR model (e.g. variance-covariance, historical simulation, or Monte Carlo) is prescribed. However, the model used must ..... be robust to adverse market environments."

[11] Artzner, P., Delbaen F., Eber, J.M., and Heath, D. (1997,1999)

[12] Joe Nocera. in *Risk Mismanagement*, *New York Times*, Jan 9, 2009. http://www.nytimes.com/2009/01/04/magazine/04risk-t.html

[13] How Caisse's bet on Quants went wrong, *Globe and Mail*, Jan 30, 2009
" Investment losses of \$25-billion or more will wipe out a huge chunk of the gains made during the Rousseau era. ....Those who did bank on it [VaR] may, like the Caisse, be suffering the consequences now. Some experts believe that VaR has caused bigger losses at many financial institutions because it has a built-in bias in favour of highly leveraged investments."

[14] "A critical Component of risk management is understanding the links between incentives and risk-taking, such as the design and implementation of compensation practices. Bonuses and other compensation should provide incentives for employees at all levels to behave in ways that promote the long-run health of the institution. " Ben Bernanke in a speech delivered May 7, 2009. www.federalreserve.cov

distribution. Then simple calculations provide

$$E(L) = E(LI(L < c)) + E(LI(L \geq c)) = \mu - \sigma\varphi(z_p) + p(\sigma z_p + \beta)$$

so if we choose

$$\sigma = \frac{\mu + p\beta}{\varphi(z_p) - pz_p}$$

then all of these loss distributions for different values of β have the same expected value 0. The trader controls the tail behavior of the distribution using leveraged products like derivatives, and the objective, reinforced by the compensation structure, is to maximize return (i.e. minimize loss). It is true that the compensation may not be a linear function of L but any monotone function will result in similar conclusions. However, this only applies until a catastrophe occurs (at which point the investor retires with her/his bonuses and/or the company is bailed out) i.e. until VaR_{1-p} is exceeded so that the objective is

$$\min_{\beta} E[L|L < c] = \frac{\mu(1-p) - \sigma\varphi(z_p)}{1-p} = \mu - \frac{\sigma\varphi(z_p)}{1-p}$$

Clearly if $\beta \to \infty$, then $\sigma \to \infty$ and the expected loss $E[L|L < c] \to -\infty$. The natural evolution of this system is toward higher returns in the center of the distribution in exchange for larger and larger losses in the remote (and usually deemed impossible) tail.

VaR is easily gamed[15], as we have seen, but there are alternative risk measures that have been proposed and are sometimes used that are much less vulnerable. For example the TailVar or Conditional Tail Expectation (CTE). The CTE, the mean of the worst $100(1-\alpha)\%$ values of the loss distribution, uses the expected loss given that it exceeds the VaR and therefore takes into account the size of the potential losses in the tail. This has the very significant advantage of coherence in the language of Artzner et al[11]. Nevertheless I would speculate that its universal adoption as a risk measure may make gaming more difficult, but not impossible. Arguably, there is no single low-dimensional measure of risk whose adoption as a constraint for traders, business lines or corporations will forestall excessive risk taking. The simple reason is that the diversity of modern potential investments, derivatives, mechanisms which provide significant leverage, and borrowing allow the construction of essentially any shaped risk distribution desired. This permits taking excessive risks (in order to achieve generally higher returns) while holding any finite dimensional functional of the distribution (the risk measures) constant. While this argument is phrased in a single dimension, the problem in the real world is made much more complicated by the high dimensionality of the potential investment universe, the evolution of the system over time, and the very complex dynamically changing parameters and dependence structure, and feedback in the system.

3. Modelling Credit Derivatives

The Gaussian Copula Model

I will describe only one of two types of models, the structural model. We generally assume that there is some filtration $\mathcal{F}_t$ for which the default time τ of a firm is a stopping time. This means that $[\tau > t] \in \mathcal{F}_t$ for all t. The filtration may depend on what information investors are assumed to have at time t[16]. If the model or market information specifies the cumulative distribution function F of the random variable τ, then it would be easy to to generate a

[15] Life after VaR, Phelim Boyle, Mary Hardy, and Ton Vorst, *Journal of Derivatives* 2005; 13, 1;

[16] see Guo, Jarrow & Zeng ; Duffie & Lando. It is assumed that investors have incomplete and lagged information at discrete time points and shown that structural models can be viewed also as reduced-form

stopping time by inverse transform; $F^{-1}(U)$ where U is a $U[0,1]$ random variable, or, if you wished to generate using a standard normal random variable Z, $\tau = F^{-1}(\Phi(Z))$. The advantage of the expression which uses $N(0,1)$ random variables is that it makes it obvious how we could incorporate dependence into the default process for a number of different firms: set the default time of firm i to be

$$\tau_i = F_i^{-1}(\Phi(Z_i))$$

for a vector $Z_1, Z_2, ...$ of dependent $N(0,1)$ random variables and cdf F_i of the default time of firm i. For better or worse, this has been the industry standard Gaussian copula model proposed by D. Li and until recently was the usual procedure for pricing CDSs and CDO's by simulation. In fact the dependence structure of the $N(0,1)$ random variables Z_i was often inherited from a small number of factors, for example

$$Z_i = \rho_i M + \sqrt{1-\rho_i}\varepsilon_i \tag{3.1}$$

for independent idiosyncratic N(0,1) factors ε_i and a common $N(0,1)$ factor M, all factors unobserved. What , if anything, is wrong with this simple model? It is fairly parsimonious, but has a very rigid correlation structure motivated more by convenience than realism. The model is quite easy to implement, to calibrate or simulate from, and the effective dimensionality is easy to control by changing the number of common factors. It also permits observable covariates or additional latent variables which may effect the risk of firms in one class but not others. In general determining the behaviour of a portfolio of a credit derivatives reduces to a multivariate normal calculation. On the negative side, some recent applications of the model to market data have resulted in estimates of the copula correlation of $\rho > 1$, clear evidence of model failure. Furthermore, correlation, ambiguous at best, is a highly inadequate indication of the dependence structure (see the example below). This model does not easily permit "contagion", whereby past defaults directly influence the probabilities of future defaults. Moreover since it focusses on a single time horizon, it essentially assumes that defaults may occur at only one time point in the future

The Gaussian copula induces too much independence especially in tails[17]. It is a well-known property of the normal copula regardless of the correlation parameter that the conditional correlation among the variables, given that they are in the tail, is zero, i.e. that under extreme conditions they behave as if they were independent. This is exactly the opposite of the behavior noted in most of the recent crises, that there is a systemic risk which tends to make losses behave more similarly in times of crisis.

Example. This example shows that the correlation in a multivariate model provides inadequate information about the dependence structure to assess the risk in a portfolio of three or more firms[18]. We assume a model for three assets much like (3.1) but suppose that the systematic factor M is replaced by an indicator random variables, in particular assume that

$$Z_i = I_i(U) + \varepsilon_i, \text{ for } i = 1,2,3, \text{ where } \varepsilon_i \text{ are independent } N(0,1). \tag{3.2}$$

Here the systematic factors $I_i(U)$ are dependent Bernoulli (i.e. zero-one) random variables. Suppose $I_i(U)$ is determined as in Figure 1. A random point U is drawn uniformly from the unit square and $I_i(U) = 1$ or 0 for $i = 1,2,3$, as the point U falls/does not fall in the corresponding rectangles A_i. All three rectangles include the middle square and are assumed of equal area p. The systematic risk factor U could represent the extent to which

[17] Embrechts, P., McNeil, A., Straumann (1999)

[18] For more detailed discussion see Embrechts, McNeil and Strauman (1999)

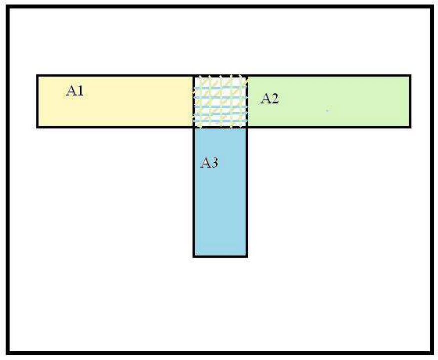

FIGURE 1. The construction of the three indicator random variables I_1, I_2, I_3. A point U is selected at random from the unit square and $I_j(U) = 1$ if U is in the rectangle A_j.

falling asset prices effect each firm and in the intersection of the three rectangles, all three firms suffer from a liquidity or asset price problem which contributes to possible default. By design, the indicator random variables $I_i(U)$ are pairwise independent but not mutually independent. To provide numbers, suppose firm i defaults if $Z_i > c$ with $c = 3$ and $p = 0.1$. The correlation between any two default indicators is 0 since $\mathrm{Cor}(Z_i, Z_j) = 0$, for $i \neq j$, identical to the independent case. The probability that a particular firm defaults is $(1-p)(1-\Phi(3)) + p(1-\Phi(2)) \simeq 0.0035$ but the probability all three firms default is

$$\left(1 - 3p + 2p^2\right)(1-\Phi(3))^3 + (3p - 2p^2)(1-\Phi(2))(1-\Phi(3))^2 + p^2(1-\Phi(0))^3 \simeq 0.00125.$$

It is hardly surprising that this is different than the answer in the independent case, but the scale of the difference is. This is around twenty nine thousand times as large as in the mutually independent case, although there are only three firms and they are "nearly" independent in that the correlation between the default indicators is 0. This discrepancy only gets worse as the number of names in the portfolio increases.

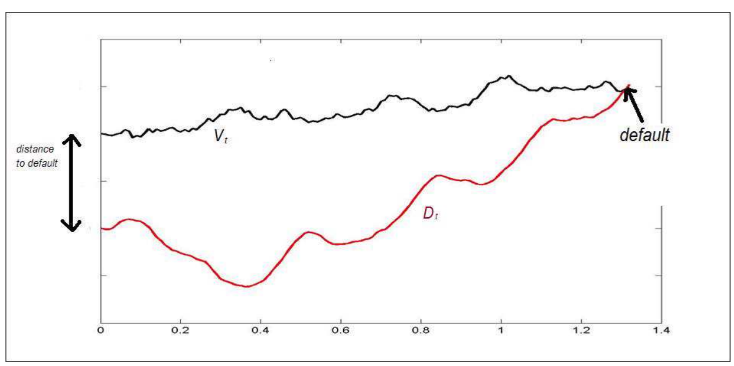

FIGURE 2. Structural model for a single firm

Dynamic Models.

The structural model. Of course another obvious fault in the Li model (3.1) is that default is only considered at some specific time horizon. A structural model attempts to link default to continuous-time processes with some intuitive appeal. In particular we model the gross market value of the firm's assets with a continuous time process V_t (this process is likely to be observed by investors only with some noise or with a time lag). In the original model due to Merton[19], $\ln(V_t)$ was modelled with a Brownian motion having variance σ^2. We also model the *Default threshold* D_t as another stochastic process. Intuitively D_t is the value of all liabilities at time t and it is typically modelled either as a deterministic function, a diffusion or an integrated diffusion. The firm defaults at time $\tau = inf\{t; V_t \leq D_t\}$, (see Figure 2). Again the processes V_t, D_t may be modelled with covariates, latent systematic factors shared among firms, and/or missing, discrete-time or lagged information[20].

Multivariate Case:

This is easily extended to the multivariate case. In general if we have N names, we can model $V_t^{(i)}$, the firm value of firm i at time t using a diffusion (or, more simply, a (Geometric) Brownian motion)

$$dV_t^{(i)} = \mu^{(i)}(t, V_t^{(i)})dt + \sigma^{(i)}(t, V_t^{(i)})dW_t^{(i)}, i = 1, 2, ..., N \tag{3.3}$$

where the Brownian motion processes $W_t^{(i)}$ are possibly correlated, and again the default barrier $D_t^{(i)}$ for firm i is either a stochastic or deterministic function of t. The usual model requires that $\mu^{(i)}, \sigma^{(i)}$ are either constants or constant multiples of $V_t^{(i)}$ so that firm value follows a (geometric) Brownian motion. In this case, to provide for the observed dependence among default times for firms, we need either to build dependence through the default barrier or generate $W_t^{(i)}$ as correlated Brownian motion. Then the default time of name i is first passage time of $V_t^{(i)}$ to its default barrier $D_t^{(i)}$.

There is a strong intuitive appeal to structural models. They appear to reflect a simple view of the way in which defaults occur. They permit using observable covariates such as book values, equity prices, leverage and debt, either in the firm value process $V_t^{(i)}$ or in the barrier process $D_t^{(i)}$. There are computational difficulties, but many of these can be partially overcome. Only in a few special cases, for example for Brownian motion in one or two dimensions with a linear boundary, are there closed form joint distributions for hitting times. The calibration to market data is difficult, especially since the dimensionality of the problem N is often 125 or more. The simplest version of the model in which both $V_t^{(i)}$ and $D_t^{(i)}$ are continuous (for example the Merton model for which $V_t^{(i)}$ is a Brownian motion and $D_t^{(i)}$ is a linear boundary) does not fit observed market data well since in the model, defaults are predictable. i.e. the default rate near $t = 0$ is zero. This can be remedied by allowing jumps[21] and/or a delayed filtration. To model default *contagion*, whereby past defaults directly influence the probability of future defaults, we would need to add jumps in the process for firm i with intensity which increases as a result of immediately preceding defaults of related firms. There is also work allowing regime-specific dependence of default rates on stock returns and volatility[22].

An Alternative Structural Model;

Achieving default dependence by forcing the driving Wiener processes to be correlated makes simulation of models of the form (3.3) much more difficult. If we were to assume independence of the driving Brownian motions,

[19] R. C. Merton. (1974)

[20] Merton; Black and Cox; Zhou; default of a company at first time when the firm-value falls below default boundary.

[21] Duffie and Lando; Guo, Jarrow and Zeng

[22] Alexander, C. And Kaek, A. (2008) Regime dependent determinants of credit default swap spreads. *J. Bank. Fin.* 32. 1008-1021

then a simpler model of the form

$$dV_t^{(i)} = \mu^{(i)}(M_t)dt + g^{(i)}(\sigma_t)dW_t^{(i)} \tag{3.4}$$

is relatively easier to simulate provided we can do so conditionally on the processes M_t, σ_t. The factors M_t, and σ_t are assumed to be market factors which simultaneously drive firm values in similar directions and increase their volatility so they, rather than the correlation among the noise processes $W_t^{(i)}$ generate dependence in the default times. Metzler and McLeish[23] assume that M_t and σ_t each satisfy a mean-reverting (CIR-like) diffusion relationship so that $\mu^{(i)}(M_t) = \beta^{(i)} M_t$, and $g^{(i)}(\sigma_t) = \xi_t^{(i)}\sigma_t$ where $\beta^{(i)}$ and $\xi^{(i)}$ are constant parameters. Then

$$\begin{aligned} dM_t &= \kappa_M(\theta_M - M_t)dt + h(M_t)dW_t^{(M)}, \text{where} \\ h(x) &= 1 + a|x-\alpha| + b(x-\alpha) \end{aligned}$$

It is relatively easy to add jumps to permit non-zero yield spreads (failure rate) at $t = 0$ and model default contagion, for example:

$$dV_t^{(i)} = \mu^{(i)}(M_t)dt + g^{(i)}(\sigma_t)dW_t^{(i)} + \delta_i(\text{jump size at } t)$$

The major advantage in the model (3.4) is that it is much easier to simulate, since conditionally on the processes M_t and σ_t, the firm values $V_t^{(i)}$ and hence their corresponding default times can be generated independently.

It is not difficult to show that first passage of the firm value (3.4) to a constant is equivalent to using a first passage time of a time-changed Brownian motion, with stochastic time change

$$T(t) = \int_0^t \sigma_t^2 dt$$

and a barrier at the integrated market process

$$D_t = d_0 + \int_0^t M_s ds.$$

Thus, conditionally on these processes M_t and σ_t^2 we can obtain virtually exact simulations of the default times $\tau^{(i)}$ either by using a linear approximation to the barrier or by using the exact simulation methods of Beskos et al.[24] or DiCesare and Mcleish[25]. This model calibrates well across different tranches/maturities for data obtained both before and after the beginnings of the current financial crisis[26].

Reduced-form (hazard rate) Models

In a reduced-form or hazard rate model, defaults are generated according to a non-homogeneous Poisson process whose intensity process (the risk factor process) is a stochastic

[23] A. Metzler, A. and McLeish, D. *A Multiname First-Passage Model for Credit Risk,* http://www.watrisq.uwaterloo.ca/Research/2009Reports/09WatRISQReports.shtml

[24] A factorisation of Diffusion Measure and Finite Sample Path Constructions. (2008) Beskos, A., Papaspilopoulos, O. and Roberts, G.O. Methodol Comput. Appl. Probab. 10, 85-104

[25] DiCesare, G and Mcleish, D.L. (2008)

[26] Multivariate First-Passage Models for Credit Risk, (2008) Thesis, University of Waterloo. A. Metzler uwspace.uwaterloo.ca/bitstream/10012

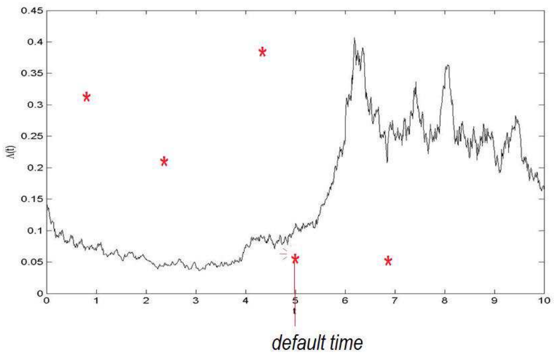

Figure 3. Generating a default in the reduced form model

process. In this case the probability of a default by a given firm in a small time interval of length Δt is

$$P[\text{default in } (t, t+\Delta t)] = \Lambda(X_t)\Delta t$$

where we may model the process X_t which controls the intensity of defaults with a jump diffusion:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \delta(\text{jump size at } t)$$

There are many papers which take this approach, and then tie the intensities of various firms together with assumptions leading to default clustering and contagion[27]. To provide some tractability to the model it is common to require that the jump sizes are independent identically distributed and that the functions driving the process X_t are affine so that $\mu(x) = K_0 + K_1 x$, $\sigma^2(x) = H_0 + H_1 x$ and $\Lambda(x) = \Lambda_0 + \Lambda_1 x$. As is often the case with such linearity assumptions, there is little empirical evidence for or against their validity.

We can visualize the simulation of defaults from a reduced form model using a graph of the intensity function as in Figure 3. We generate a Poisson process with constant intensity 1 in a region containing this graph, and then accept the time coordinate of the first point below this graph as a default time as shown.

There is to some degree a difference between structural models and the reduced form models here in the way in which defaults are triggered. Does default clustering occur because in certain periods, the intensities of default are influenced by a third common factor, does one default direclty influence another, or there are correlated processes which may represent something like firm value that are approaching default barriers at roughly the same times? These are close to classical problems in statistics, in distinguishing causal relationships from more general correlation. The network models of the next section provide a simple attempt at a more realistic structure.

4. Networks and Leverage Multiplication

Current financial models used for pricing derivatives and assessing risk are usually simple models with an exogenous input and an output that consists either of an asset value or a

[27]e.g. Jarrow and Turnbull (1992); Longstaff and Schwartz (1995), Hull and White (2001), Gieseke (2008), Litterman and Iben (1991); Madan and Unai, (1998), Duffie and Singleton (1999)

measurement of risk. Similar simple models may be used to assign portfolio weights for a hedge fund.

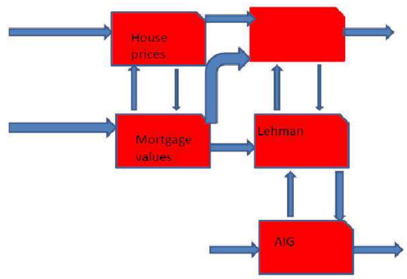

FIGURE 4. A fragment of a financial network

In practice the financial system is an incestuous relationship among banks, investors, brokerage houses, hedge funds, rating agencies, and governments. The slings and arrows of fortune of firm A are products of competition, predatory trading, counterpary risk, liquidity, credit, investment whims and returns of most other major players in the marketplace, quite apart from the vagaries of consumer demand. Liquidity constraints and margin requirements may force fire sales of assets, generating a "liquidity spiral". In short, the usual assumption in finance of exogenous sources of noise, doubtful at the best of times, is potentially highly misleading under crisis conditions. Of course the financial network is an exceedingly complex, and to date largely opaque object, but a fragment of such a network might more resemble Figure 4 in which there are multi-way interactions between a large number of counterparties. For illustration, consider the much simpler network in Figure 5.

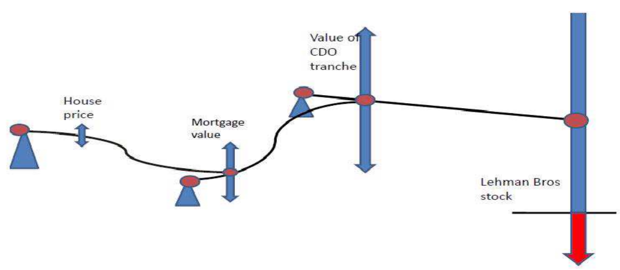

FIGURE 5. A network with three sources of amplification. Small changes in house prices effect larger changes in the value of mortage on these houses. These result in larger changes in the value of a given tranche of a CDO. Becasue of a low equity/asset ratio, this results in even larger changes to the price of a stock.

The purpose of this figure is to show the leverage multiplication effect that can occur with a small number of products or firms in a series. This would seem to imply that the equity/asset ratio of around 4% for the major investment banks in 2006 constitutes a significant underestimate of the total leverage in the financial system, and the vulnerability of the system to deterioration of circumstances. There is an excellent discussion of the growth and stability of financial networks in the talk by Haldane[28].

[28]RETHINKING THE FINANCIAL NETWORK, Andrew G Haldane
http://www.bankofengland.co.uk/publications/speeches/2009/speech386.pdf

Example: Simulation of a two-node network.

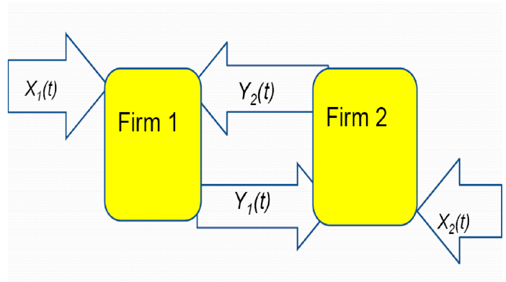

FIGURE 6. A simple Network with two nodes.

In an effort to investigate the stability of such networks when there is both significant randomness and endogenous factors, we simulate the two-note system displayed in Figure 6. Each node (a simple hedge fund) is permitted to invest in one of four investments at any given point of time, so for example Firm 1 chooses weights on the risk-free asset, shares of the exogenous stock $X_1(t)$, shares of the endogenous stock $Y_2(t)$ and an approximately at-the-money call option on the endogenous stock $Y_2(t)$ whose value, $BS(Y_2(t))$ is determined by the Black Scholes formula. If this option is well out of the money (so its value is less than 5% of the value of the corresponding asset) then no weight is permitted on the option. The investment opportunities of Firm 2 are the mirror image of those for Firm 1. Each firm acts each day to maximize the risk-adjusted return. The return to stock 1 on day t, $R_1(t)$ is measured as

$$R_1(t) = w_1(t) \times \text{Return}(X_1(t)) + w_2(t) \times \text{Return}(Y_2(t)) + w_3(t) \times \text{Return}(BS(Y_2(t)))$$

where the weights on day t , $w_1(t), w_2(t), w_3(t)$, and $w_4(t) = 1 - w_1(t) - w_2(t) - w_3(t)$ are constrained to add to one and

$$\text{Return}(X_1(t)) = \frac{X_1(t) - X_1(t-1)}{X_1(t-1)}.$$

Values can, however, be negative to indicate shorting or borrowing. For simplicity we assume all values are expressed in discounted terms so that the return on the risk-free asset, r is 0. Then the weights $w_i(t)$ are chosen to maximize risk-adjusted returns as measured by the Sharpe ratio,

$$\frac{E(R_1(t))}{\sqrt{Var(R_1(t))}}$$

subject to a constraint on the daily VaR, namely the value at risk (the maximum daily loss with confidence level 99%) calculated from the daily returns less than -4%. Here both the mean and covariances of $(X_1(t), Y_2(t), BS(Y_2(t)))$, required for the calculation of the Sharpe ratio, were estimated using a historical exponential reweighted moving average as might be done in practice. Specifically we update the mean $mean(t) = E(R_1(t))$ estimated each day using using a weighted average of the estimate from day $t-1$ and the sample mean obtained from the observations collected on day t.

$$\text{mean}(t) = 0.9 \times \text{mean}(t-1) + 0.1 \times \text{sample mean}$$

We use the same strategy for estimating the covariances matrices at each of the two nodes. By varying the parameters of the simulation, including the drift and the volatility of the exogenous asset $X_j(t), j = 1, 2$ both independent geometric Brownian motion processes, and recording the frequency with which the system experiences either a bubble (defined here as a doubling of a value) or a bust (defined as a 50% reduction in value) we are able to discover the sensitivity of the network to parameters such as the underlying volatilities and any imposed constraints on the weights.

Our simulation[29] results assume that the return on the exogenous stocks $X_j(t)$ were (in real terms) 5% per annum and that each of the hedge funds removed a management expense ratio at a rate of 5% per annum so that the system as a whole might be expected to be in approximate equilibrium. We use a time horizon of 1 year and values of the volatility in the interval $0.1 \leq \sigma \leq 0.5$. The probability of either a bubble or a bust is remarkably insensitive to the volatility fed into the system (in part because we may have compensated for this volatility by adjusting the daily VaR) and the probability of one or the other within one year is around 20% (see Table 1). This although we constrained the daily Value at Risk so that under the standard assumptions of independence, the probabability of a bust in a year should be around 1%.

Table 1: Simulated probabilities of bubble or bust for two-node network

σ	Prob bubble	Prob bust	Total
.1	0.091	0.099	0.200
.2	0.120	0.080	0.200
.3	0.118	0.078	0.196
.4	0.113	0.091	0.204
.5	0.096	0.094	0.190

Suppose we add the additional constraint that no weight is permitted to be below -1 or above 1. This corresponds roughly to a margin constraint. Does this dramatically effect the proportion of booms and busts in the above model? Interestingly, although there appears to be a slightly higher probability of a bubble, the probability of a bust is scarcely affected and the total remains in the same ballpark, around 20%

What happens if we increase the number of nodes? Obviously there is an upper limit to the number that permit fast computation so that the simulations are feasible, but we did repeat this experiment with 3 nodes, so that each of the three firms have a total of five potential risky investments (stock in each of the other two firms, a call option on that stock, and one exogenous and independent stock). In addition, of course, there is the risk-free account. In this case, in order to partially compensate for the increased number of firms, we changed the daily VaR constraint, requiring that the 99.9% daily VaR be less than 2%. This would imply, under the usual assumption of independent daly returns, that the probability of a bust or 50% decrease for a particular firm in a one year is about 2×10^{-11} an event that occurs only once every 500 billion years and not completely unlike the market in 2007[30]. Simulation results in this case[31] indicate that instead of the 10-fold increase in probability we saw in the two- node case, we experience a hundred-billion-fold increase in in the probability of either bubble or bust over the case in which we assume independent daily returns, from around 2×10^{-11} to about 0.4.

[29] I am grateful to Kaushiki Bhomick for help with these simulations

[30] "We were seeing things that were 25-standard deviation moves, several days in a row," David Viniar, Goldman's chief financial officer, August 13, 2007.

http://www.ft.com/cms/s/0/d2121cb6-49cb-11dc-9ffe-0000779fd2ac.html?nclick_check=1

[31] These results are **highly sensitive** to the parameters: in this case based on 2000 simulations, a MER of 2% per annum and 30 observations/day. We prevented investing in far out-o-the money calls (call value<.05*asset value) and to try to imitate margin limits, truncated weights to the interval $[-0.8, 1]$)

5. Conclusion

There are many lessons in the current financial crisis relevant to the use of statistical methods in financial modelling. One complicating factor to choosing and fitting a model is the Google effect: the speed of information flow is now faster by orders of magnitude than it was even several years ago, so that effects, including feedback, are felt more rapidly and globally than ever before. Even in a simple network, feedback can dramatically amplify the probabilities of either a crash or a bubble by factors of billions.

Statistical models in finance need be dynamically changing to reflect financial innovation, changing volatility and correlation, feedback under extreme conditions, and leverage effects. Multivariate models should to be chosen, *not for analytical tractability*, but with a view of the actual interdependence of the firms. Moreover, there needs to be sufficient statistical literacy among the consumers of risk measures to assess the legitimacy of the model assumptions and the likelihood of those scenarios which result in extreme losses. *Risk is not a number.*

The majority of wide-tailed multivariate distributions show very different dependence in the extremes than in the center of the distributions whereas the multivariate normal has essentially independent extremes regardless of the correlation parameter[32]. In practice, the high degree of tail dependence, partially a product of investor psychology and fear, is more consistent with these wide-tailed distributions than with the multivariate normal, and modelling this dependence is essential in an adequate model for credit risk. In a structural model, is it reasonable to assume that the firm-value process of a firm near default is essentially the same as that for a healthy firm, when fear and predation creates enormous swings in a stressed market? Do we use the same model for the health of an infirm octogenarian and a 21 year-old Olympian?

Edmund Phelps, who won the Nobel prize for economics in 2006, is highly critical of today's financial services. "Risk-assessment and risk-management models were never well founded," he says. "There was a mystique to the idea that market participants knew the price to put on this or that risk. But it is impossible to imagine that such a complex system could be understood in such detail and with such amazing correctness... the requirements for information... have gone beyond our abilities to gather it."[33] From a modeler's perspective, this may be an overly pessimistic view, but those drunk on the profits and well-worn models of the past will surely find these words and the events since 2007 sobering.

References

[1] Alexander, C. And Kaek, A. (2008) Regime dependent determinants of credit default swap spreads. *J. Bank. Fin.* 32. 1008-1021

[2] Artzner, P., Delbaen F., Eber, J.M., and Heath, D. (1997). Thinking Coherently; *Risk* 10, 68-71

[3] Artzner, P., Delbaen F., Eber, J.M., and Heath, D. (1999) Coherent Measures of Risk; *Mathematical Finance*, 9, 203-228

[4] Azizpour, S. and Giesecke, K. (2008) Self-Exciting Corporate Defaults: Contagion vs. Frailty (working paper)

[5] Basak, S. and Shapiro, A. (2001) Value-at-Risk-Based Risk Management: Optimal Policies and Asset Prices. *Rev. Fin. Stud.* 14. 371-405

[6] Basel Committee on Banking Supervision International Convergence of Capital Measurement and Capital Standards. (2005) A Revised Framework.

[32] Embrechts, P. Klüppelberg, C. and Mikosch, T. (2003)

[33] In Plato's Cave. *The Economist*, Jan 22, 2009

[7] Black, F. And Cox, J. (1976) Valuing Corporate Securites: Liabilities: Some Effects of Bond Indenture Provisions *J. Fin.* 31. 351-367.
[8] Boyle, P., Hardy, M. And Vorst, T. (2005) Life After Var. *Journal of Derivatives* 13,1, 48-55.
[9] Crouhy, M.G., Jarrow, R.A. and Turnbull, S. M. (2007) The Subprime Credit Crisis of 07 (working paper)
[10] DiCesare, G. and McLeish, D.L. (2008) Simulation of Jump Diffusions and the Pricing of Options. *Ins. Math. Econ.* 43, 316-326
[11] Duffie, D. and Singleton, K. J. (1999) Modeling the term structures of defaultable bonds. *Rev. Financial Studies*, 12:687–720
[12] L'Ecuyer, P. and Simard, R. (2008) TestU01: A C Library for Empirical Testing of Random Number Generators, http://www.iro.umontreal.ca/~simardr/testu01/tu01.html
[13] Embrechts, Paul, Lindskog,Filip and Alexander McNeil (2001) Modelling Dependence with Copulas and Applications to Risk Management
[14] Embrechts, P., McNeil, A., Straumann, D. (1999) Correlation: Pitfalls and alternatives *RISK*, May, 69-71
[15] Embrechts, P. Klüppelberg, C. and Mikosch, T. (2003) *Modelling Extremal Events for Insurance and Finance,* Springer-Verlag, 1st ed.
[16] Giesecke , K. (2008) An Overview of Credit Derivatives (working paper)
[17] Guo, X., Jarrow, R.A., and Zeng, Y. (2008) Credit Risk Models with Incomplete Information. (working paper)
[18] Haldane, Andrew, G. (2009) Rethinking the Financial Network http://www.bankofengland.co.uk/publications/speeches/2009/speech386.pdf
[19] Hull, J. and A. White, A. (2001) Valuing credit default swaps II: modeling default correlations. *Journal of Derivatives*, 8(3):12–22
[20] Jarrow, R.A. and S. M. Turnbull (1995) Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50:53–86
[21] R. Litterman and T. Iben. (1991) Corporate bond valuation and the term structure of credit spreads. *Financial Analysts Journal*, Spring: 52–64
[22] Longstaff, F. and Schwartz, E. (1995) Valuing risky debt: a new approach. *Journal of Finance*, 50. 789–820,
[23] D. Madan, D. and H. Unal (1998) Pricing the risks of default. *Review of Derivatives Research*, 2:121–160
[24] R. C. Merton. (1974) On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29:449–470
[25] Metzler, A. and McLeish, D. (2009) A Multiname First-Passage Model for Credit Risk. (working paper at http://www.watrisq.uwaterloo.ca/Research/2009Reports/09WatRISQReports.shtml)
[26] Morris, C. (2008) *The Trillion Dollar Meltdown* PublicAffairs, New York
[27] Nocera, Joe (2009) *Risk Mismanagement* in the *New York Times*, Jan 9, 2009.

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1